The Ways to Help Students Better Understand Taylor Mean Value Theorem

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ABSTRACT
This article attempts to use common cosine functions and exponential functions as examples to explain Taylor mean value theorem through graphical examples and specific numerical explanations, making abstract content more intuitive and achieving good results in practical teaching.

Keywords: *Taylor mean value theorem, Lagrangian remainder, Peiano type remainder, Graphic teaching method.*

1. INTRODUCTION

As a further extension of Lagrange mean value theorem, Taylor's mean value theorem not only holds an important position in theory, but also has important applications in approximate calculations, series theory, determining the concavity and convexity of curves, proving equations and inequalities, median problems, determinant calculations, error estimation, and solving differential equations. It can be said that most of the problems that can be solved by the differential mean value theorem can be solved by the Taylor mean value theorem. The Taylor mean value theorem is the foundation of analytic function theory, helping us to decompose complex functions into simple polynomial sums and use polynomial approximations with finite terms to study complex functions. It can be said that Taylor mean value theorem is the pinnacle of univariate differentiation ([1]). For such an important theorem, students face significant challenges in their learning. Teachers who have taught advanced mathematics know that if Taylor mean value theorem is explained using the usual blackboard teaching method, no matter how hard the teacher tries to explain it, students will be confused. Not only will the proof process of the theorem be difficult to accept, but they will also have almost no idea what the theorem answers. Whenever Taylor's formula is mentioned, students shake their heads and show a fear of difficulty. Some students have even lost confidence in learning advanced mathematics as a result.

Teachers have always been familiar with epistemology, from individual to general, from concrete to abstract, from sensibility to rationality, which is an upward movement of cognition, even called a leap in cognition. This is not only correct, but also important. The late famous Chinese mathematician Hua Luogeng once said that moving from concrete to abstract is an important path in the development of mathematics. Abstraction is often said to be the most fundamental characteristic of mathematics. Helping students better understand and master abstract mathematical concepts and theories is a fundamental task in mathematics teaching.

In order to achieve better teaching results, many teachers have made beneficial attempts in the teaching of Taylor mean value theorem from various aspects. For example, in reference [2], the author linked function approximation with a super imitation show in real life. In reference [3], the author provided new proof methods different from the textbook to enhance students' understanding of Taylor mean value theorem. In order to enhance the intuitiveness of teaching content and deepen students' understanding of the learned knowledge, the author used Matlab's drawing function to assist teaching in reference [4].

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In this article, the author will not delve deeper into the theoretical exploration of Taylor's mean value theorem. Instead, the author will use the drawing and calculation functions of Matlab mathematical software to explain Taylor mean value theorem through graphics and specific data.

2. UNDERSTANDING TAYLOR'S FORMULA FROM A GEOMETRIC INTUITIVE PERSPECTIVE

To understand Taylor's formula from a geometric perspective, teachers can first start with the differentiation of functions. Before learning Taylor's formula, students have already learned the concept of differentiation and know that if a function $f(x)$ is differentiable at the point x_0, there is an approximate calculation formula $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, that only makes sense when $| x - x_0 |$ is quite small. From the image, it appears to be in the vicinity of the point $(x_0, f(x_0))$, using the tangent $f(x) = f(x_0) + f'(x_0)(x - x_0)$ of the point $(x_0, f(x_0))$ on the curve $y = f(x)$ to approximate the curve $y = f(x)$. However, from a geometric perspective, it is evident that if the function originally representing the curve has good properties, such as, if there is a derivative up to the order $n+1$ near x_0, using the curve to approximate instead of the curve will result in better performance (with smaller errors). But what curve should be used to approximate the original curve? Firstly, the selected curve should be at least simpler than the original curve. Polynomial functions only perform finite addition, subtraction, and multiplication operations on independent variables and constants, and polynomial functions have any order of derivative at any point. Therefore, it is ideal to approximate the original curve with a polynomial function. Secondly, the selected curve should have certain similar characteristics to the original curve at point x_0, such as passing through $(x_0, f(x_0))$. That is to say, there is no error at point x_0 of the polynomial sum $f(x)$. Polynomial functions and f (x) have the same tangent at point $(x_0, f(x_0))$, the same concavity and convexity at point $(x_0, f(x_0))$, the same degree of curvature at point $(x_0, f(x_0))$, and so on. And these characteristics of the curve at point x_0 are precisely determined by the function value of function $f(x)$ at point x_0 and the values of the various derivatives of $f(x)$ at point x_0. Taylor's theorem answers the question that if function $f(x)$ has a derivative up to $n+1$ order near x_0, then function $f(x)$ can be approximated by a n degree polynomial and can provide a specific estimate of the error. The Taylor mean value theorem consists of several Taylor formulas.

2.1 Taylor Formula with Lagrange Residue

Theorem 1: If the function $f(x)$ is a continuous derivative that exists up to $n+1$ order in a certain neighborhood of x_0, there must be ξ between x and x_0, so that the following equation holds:

$$f(x) = [f(x_0) + \frac{f^{'}(x_0)}{1!}x + \frac{f^{''}(x_0)}{2!}x^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}x^n] + R_n(x)$$
$$= P_n(x) + R_n(x), (1)$$

$R_n(x) = \frac{f^{(n+1)}(\xi)(x - x_0)^{n+1}}{(n+1)!}$ (1) is called the n Taylor formula with Lagrange residue, where $R_n(x) = \frac{f^{(n+1)}(\xi)(x - x_0)^{n+1}}{(n+1)!}$ is referred to as the Lagrange residue.

At $x_0 = 0$, equation (1) becomes:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$
$$= P_n(x) + R_n(x), (2)$$

The formula (2) is an n order McLaughlin formula with Lagrange residues.

2.2 Taylor's Formula for the Peiano Type Remainder Term

Theorem 2. ([5,6]) If the function $f(x)$ has a derivative of n order at x_0, then

$$f(x) = [f(x_0) + \frac{f'(x_0)}{1!}x + \frac{f''(x_0)}{2!}x^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}x^n] + o(x - x_0)^n,$$
$$= P_n(x) + o(x - x_0)^n, \text{ (2).}$$

The formula (2) is a Taylor formula of n order with a Peano type remainder at point x_0 of $f(x)$, where $o((x - x_0)^n)$ is called a Peano type remainder. When $x_0 = 0$, equation (2) becomes:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n), (3)。$$

The formula (3) is an n order McLaughlin formula with Lagrange residues. If $f(x) = \cos x$, equation (3) becomes:

$$\cos x = [1 - \frac{1}{2!}x^2 + \frac{x^4}{4!}\cdots + \frac{(-1)^{(n)}}{(2n)!}x^{2n}] + o(x)^{2n+1}$$
$$= P_{2n}(x) + o(x)^{2n+1}。$$

In “Figure 1”, the author has drew images of $f(x) = \cos x$ and $P_{2n}(x), n = 3,4,5$ from the same coordinate plane.

If $P_{2n}(x), n = 3,4,5$ is used to approximate $f(x) = \cos x$, there are relative errors at several specific points.(“Table 1”)

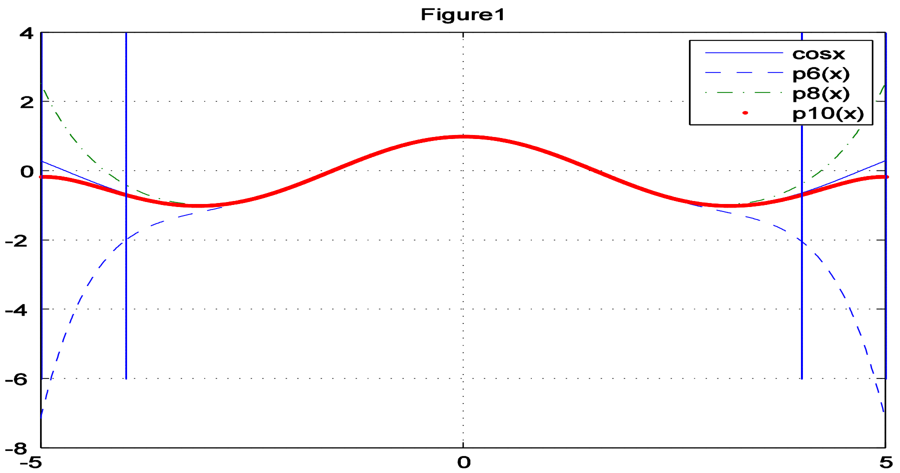

Figure 1 Images of $f(x) = \cos x$ and $P_{2n}(x)$, $n = 3,4,5$ from the same coordinate plane

Table 1. Relative errors at several specific points

x	1	Relative error at $x=1$	2	Relative error at $x=2$	4	Relative error at $x=4$
$\cos(x)$	0.5403		0.4161		-0.6536	
$p_6(x)$	0.5403	4.5399e-005	0.4222	0.0144	-2.0222	0.6768
$p_8(x)$	0.5403	5.0619e-007	-0.4159	6.58424-004	-0.3968	0.6472
$p_{10}(x)$	0.5403	3.842e-009	-0.4162	2.0104e-005	-0.9910	0.0469

a Relative error=| approximate value - exact value/approximate value | $4.5399e-005 = 4.5399*10^{-5}$, otherwise the same

From the above "Table 1", it can be seen that for the same $P_{2n}(x)$, $n = 3,4,5$, the smaller $|x|$, the smaller the relative error. For the same x, the larger the n in $P_{2n}(x)$, $n = 3,4,5$, the smaller the relative error. In fact that and have for a given x ($|x|$ does not need to be small), as long as n is sufficiently large, the relative error generated when using $P_{2n}(x)$ to approximate $f(x) = \cos x$ can be arbitrarily small. This conclusion is very profound, that is, at any point, a polynomial function can be used to approximate a periodic function $f(x) = \cos x$, and the error can be arbitrarily small.

If $f(x) = e^x$, equation (1) becomes

$$e^x = 2 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \frac{e^{\xi} x^{n+1}}{(n+1)!},$$

while ξ is located between x and 0.

From the above equation, it can be seen that for a given x, as long as n is large enough, when using $p_n(x)$ to approximate e^x, the error is less than $\frac{e^x x^{n+1}}{(n+1)!}$. (Due to $\lim_{n\to\infty} \frac{e^x x^{n+1}}{(n+1)!} = 0$, as long as n is large enough, the error can be arbitrarily small.)

In "Figure 2", the images of e^x and $P_n(x)$, $n = 4,6,7,9$ are drawn from the same coordinate plane.

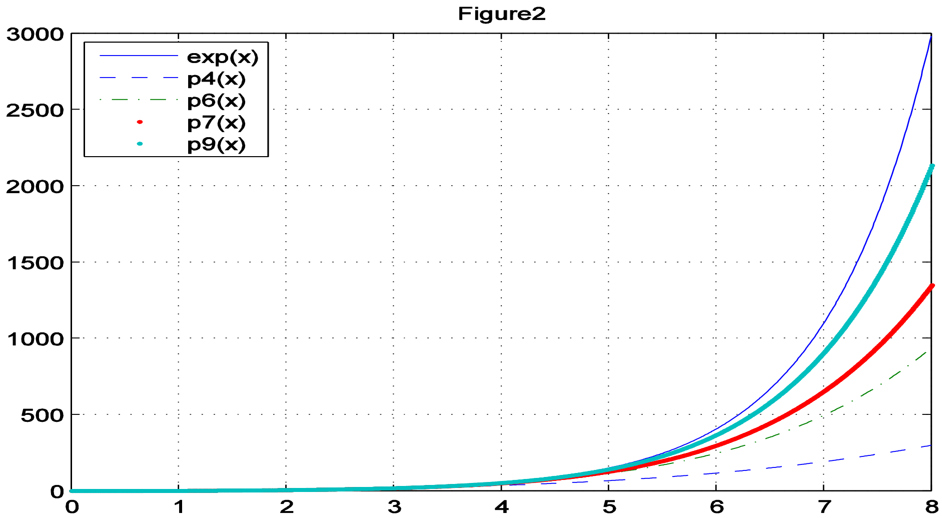

Figure 2 Images of e^x and $P_n(x)$, $n = 4,6,7,9$ from the same coordinate plane.

It still depends on the relative error when $P_n(x)$, is used to approximate e^x at several special points.

Table 2. Relative errors at several specific points

x	2	Relative error at $x=2$		5	Relative error at $x=5$	8	Relative error at $x=8$
e^x	7.3891			148.4132		2.9810e+003	
$p_4(x)$	7	0.0556		65.3750	1.2702	297	9.0369
$p_6(x)$	7.3556	0.0046		113.1181	0.3120	934.1556	2.1911
$p_7(x)$	7.3810	0.0011		128.6190	0.1539	1.350e+003	1.2077
$p_9(x)$	7.3887	4.6500e-005		143.6895	0.0329	2.1362e+003	0.3907

From the above "Table 2", it can be seen that for a given x ($|x|$ does not need to be small), as long as n is sufficiently large, the relative error generated when using $p_n(x)$ to approximate e^x can be arbitrarily small.

In addition to the Taylor formulas with Lagrange residues and Peano residues given above, there are also Taylor formulas with integral residues and Taylor formulas with Cauchy residues ([7]). The other two formulas have a wide range of applications in mathematics, revealing deeper mathematical content while becoming more complex. As teachers are teaching freshmen, they will not discuss them.

3. CONCLUSION

People always hope to simplify complex problems. For a complex function, they always hope to approximate it with a simple function. Polynomial functions only perform finite addition, subtraction, and multiplication operations on the independent variables. Polynomial functions are both simple and have arbitrary derivatives at any point. Taylor mean value theorem states that if a function has good properties ($f(x)$ has a derivative up to $n+1$ order at every point on the open interval containing x_0), $f(x)$ can be approximated by a polynomial of degree. The polynomial used to approximate $f(x)$ is uniquely determined by $f(x)$, and as long as n is large enough, the error can be minimized. Polynomial functions are functions that students are very familiar with in high school, such as linear functions and quadratic functions. The curves of these functions leave a deep impression on every student's mind.

In this article, the author made a beneficial attempt to teach Taylor mean value theorem, using graphic demonstrations that are easy for students to accept and using concrete polynomials to approximate two familiar functions to explain Taylor's mean value theorem, so that students can elevate from concrete to abstract level on the basis of intuitive understanding.

The graphics here are generated using the mathematical software Matlab, and the interface between PPT and Matlab can be easily switched through hyperlinks and other methods. Therefore, with a little effort spent on making courseware during lesson preparation, abstract and difficult mathematical content can be made intuitive and easily accepted by students, achieving good teaching results in actual teaching.

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